The Derivative as Slope of Tangent Line

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Abstract

We examine critically the statement “ derivative measures slope of tangent line.”

1 Introduction

Most calculus textbooks stress the connection between derivatives and slopes of tangent lines, and in this regard they seem to fall into two groups: those that offer calculation of tangent line slopes as an “application” of the derivative (see, e.g., [1], pp. 31-32,) and those that define the concept of tangent line in terms of the derivative (see, e.g., [3], p. 75.)

Calculation of slopes as an application only makes sense if there is a received notion of *tangent line*, e.g., from geometry. The geometric concept has certainly existed since Euclid, but it has been somewhat fluid over time, and difficult to pin down in ancient sources. Both Euclid and Apollonius of Perga give detailed constructions of tangent lines, Euclid in the case of circles (see Propositions 16 and 17 of *Elements III*), and Apollonius for the conic sections (see Propositions 33 and 34 of *Conics I.*) Of course, neither author actually uses the phrase “tangent line”. The word *tangent* is English, and derived from the Latin *tangere,* to touch. Both ancient authors distinguish between lines that touch a curve and those that cut the curve. They also note that lines touching a curve have the property that no other line can be placed between the touching line and the curve. Using the derivative to define tangency, on the other hand, begs the question “what is this new concept of tangency good for?” I offer the term *osculating line* as a replacement for tangent line defined using the derivative, with little hope that it will be accepted.

In this paper we explore the connections between calculus and a concept of tangency strict enough to cover what the ancient geometers had in mind, but lax enough to allow for examples that go beyond the conic sections.

Throughout, we shall adopt the convention that, if we designate a point on a graph by a capital letter, then the corresponding lower case letter is the x coordinate of the point. Each non-vertical line divides the plane unambiguously into a region above the line and a region below it. More precisely, if L is an affine linear function, then a point $X(x, y)$ lies above L if and only if $y \geq L(x)$, and lies below it if $y \leq L(x)$. We say a set of points (e.g., a graph) lies above (resp. below) L if and only if each point of the set lies above (resp. below) L.

Let A be a point on the graph G of a function f. A non-vertical line L with equation $y = f(a)+m(x-a)$ is said to be a *geometric tangent line* to the graph of f at A (or at a) if the graph meets L only in the point A, and G lies either above L or below L. In cases where there is no ambiguity, we denote the tangent line at $x = a$ by L_a, and we use this same symbol interchangeably for the affine linear function and its graph. We shall denote by m_a the slope of L_a. A function is a GT function if it has a unique geometric tangent at each point of its graph.

If we dropped the requirement that the graph be on one side of L, attempting to emulate the simpler definition of tangent that works for conic sections – a non-vertical line that intersects the graph in exactly one point – we would severely restrict the class GT. For example, any line of negative slope meets the graph of the exponential function in exactly one point. Since many such lines pass through each point on the graph of $y = e^x$, this function would not belong to class GT for having too many "tangents".

2 Tangent Lines without Calculus

If a function f is known to be GT *a priori,* (for example, if its graph is a conic section,) then the slopes of tangent lines to the graph of f can often be found without calculus. For example, let m be the slope of the tangent line to the graph of $y = x^2$ at $x = a$. Since this function is GT, the equation

$$x^2 = a^2 + m(x - a)$$

has the unique solution $x = a$. This information alone is sufficient to determine m: If $x \neq a$, then

$$m = \frac{x^2 - a^2}{x - a} = x + a.$$

If m had any value besides $2a$, then $x = m - a$ would give a second solution.

As a second example, consider the problem of finding the slope of the tangent line to $y = 1/x$ at $x = a > 0$. The graph is an hyperbola and has a unique geometric tangent at each point of its domain. Thus, the equation

$$\frac{1}{x} = \frac{1}{a} + m(x - a)$$

has the unique solution $x = a$, where m, as above, stands for the slope of the tangent line at $x = a$. If $x \neq a$ then

$$m = \frac{\frac{1}{x} - \frac{1}{a}}{x - a} = -\frac{1}{ax}$$

If m had any value besides $-\frac{1}{a^2}$, then $x = -\frac{1}{am}$ would give a second solution.

Let us show that the function $f(x) = |x|^n$ is GT for each natural number $n \geq 2$. For this, we recall Bernoulli's inequality,

$$(1 + y)^n \geq 1 + ny,\ \ n \in \mathbb{N},\ \ ny \geq -1,$$

with strict inequality if $n \geq 2$ and $y \neq 0$. This can be proved by induction on n. For another quick proof, apply the AGM inequality to the set of n non-negative numbers $\{1, 1, \ldots, 1, 1 + ny\}$. Taking $y = x - 1$ gives

$$x^n \geq 1 + n(x - 1),\ \ x \geq 1 - \frac{1}{n}.$$

For all other values of x the right-hand side is negative, so the inequality

$$|x|^n \geq 1 + n(x - 1)$$

holds for all x. If $a > 0$ we obtain, after replacing x by x/a that

$$|x|^n \geq a^n + na^{n-1}(x - a), -\infty < x < \infty$$

holds for any natural number n. Also, strict inequality holds when $x \neq a$. This shows that f has a geometric tangent at each $a > 0$. By a symmetrical argument, the same is true for $a < 0$. The x-axis is clearly a geometric tangent at the origin. Since the slope, $na|a|^{n-2}$, of these geometric tangents is a continuous function of a when $n \geq 2$, the desired result follows from Theorem 3.2 below. (We note that this shows $|x|^n$ is GT without using the derivative, and thus the usual elementary power rule of calculus can be derived, as in the case $n = 2$, "without calculus".)

As a final example, let us show that the exponential function is GT. Defining e^x for irrational x is a delicate matter if one proceeds from first principles, but it can all be done without using differentiation or integration. Problem 6, p. 23 of [2] gives an outline of the steps required to define exponentials with real exponents and to establish their basic properties. The arguments use only algebra and the completeness of the real number system. Similar arguments can be used to show that the exponential function is continuous. The first step is to show that the sequence $x_n = (1+1/n)^n$ is increasing in n. Bernoulli's inequality is handy once again:

$$\frac{x_{n+1}}{x_n} \quad = \quad \left(1 - \frac{1}{n^2+2n+1}\right)^n \frac{n+2}{n+1} \quad \geq \quad \left(1 - \frac{n}{n^2+2n+1}\right) \frac{n+2}{n+1},$$

and a little algebra shows the last expression is greater than one. Next, one shows that the sequence x_n is bounded above. One way to do this is to expand $(1+1/n)^n$ and use the easy estimates

$$\binom{n}{k} \frac{1}{n^k} \quad \leq \quad \frac{1}{k!}, \quad \text{and} \quad \sum_{k=0}^{n} \frac{1}{k!} \; < \; 3.$$

It follows from completeness that x_n has a limit, and we define this limit to be the value of e, Euler's constant. Next, for $x \in \mathbb{Q}_+, x = k/m$, we have that

$$\lim_{n\to\infty} \left(1 + \frac{x}{nk}\right)^{nk} \quad = \quad \lim_{n\to\infty} \left(1 + \frac{1}{nm}\right)^{nmx} \quad = \quad e^x.$$

. By Bernoulli's inequality (again!), we have

$$\left(1 + \frac{x}{n}\right)^n \quad \geq \quad 1 + x, \quad x \geq -1,$$

and therefore

$$e^x \geq 1 + x, \quad x > -1, \quad x \in \mathbb{Q}.$$

Since e^x is positive for all rational x, the condition $x > -1$ can be dropped, and then the inequality persists for all real numbers x since both sides are continuous functions of x. Moreover, strict inequality holds for all $x \neq 0$. (One may check, e.g., that $(1 + x/n)^n$ increases strictly in n for every $x \neq 0$.) Finally, for any real numbers a and x, we have

$$e^x = e^a e^{x-a} \geq e^a(1 + (x-a)) \;=\; e^a + e^a(x-a),$$

with strict inequality unless $x = a$. This shows that the exponential function has a geometric tangent at each point A on its graph, with slope equal to e^a. Since the function giving the slope is continuous, it follows from Theorem 3.2 below that the exponential function is GT.

3 Which Functions are GT?

In this section we determine exactly which functions are GT, and provide a useful criterion for functions with geometric tangents to have unique geometric tangents.

Theorem 3.1. *A function f is GT if and only if it has a continuous, strictly monotone derivative.*

Suppose f has geometric tangents but is not yet assumed GT. Suppose A and B are distinct points on the graph, G, of f. We shall argue first that G must lie on the same side of L_a and L_b. Suppose in fact it were the case that G is above L_a and below L_b. If $m_a \neq m_b$ then L_a and L_b intersect in a point C. Since G is below L_b and C is on L_b, we have $f(c) \leq f(b) + m_b(c-b) = L_b(c)$, and since G is above L_a and C is also on L_a, we have $L_a(c) = f(a) + m_a(c-a) \leq f(c)$. Since C lies on both lines, we have indeed $L_b(c) = L_a(c) = f(c)$. Since C must be distinct from at least one of A and B, it follows that G meets at least one of the lines in more than one point, contrary to hypothesis.

There remains the possibility that L_a and L_b are parallel, with G lying entirely within the strip between them. Assume for definiteness that $a < b$ and L_b is above L_a. (Other cases are handled similarly.) In this case, if we pick any $a < c < b$, the same argument shows that $m_c = m_a = m_b$: since G lies on one side of L_c

or the other, and on opposite sides of L_a and L_b, we may pair one of the two lines with L_c in the argument given. Since point A is below L_c, all of G is below L_c. Since point B is above L_c, all of G is above L_c. This is only possible if the three lines coincide, and this is not possible, since A and B are distinct.

If $A = B$ and we had two distinct tangents at A, L_1 and L_2, then neither lies on one side of the other. No point of G, apart from A itself, can lie between the two lines, since G must lie on one side of each. Thus G lies on the same side of L_1 and L_2.

We have shown, therefore, that G is either above all of its tangent lines, or below all of its tangent lines. Suppose it is the former. We shall show, in this case, that the mapping $b \to m_b$ is strictly increasing. Let $a < b$. Since G lies above L_b we have

$$0 \leq f(x) - f(b) - m_b(x-b), -\infty < x < \infty$$

Similarly,

$$0 \leq f(x) - f(a) - m_a(x-a), -\infty < x < \infty$$

In particular, we have

$$0 < f(a) - f(b) - m_b(a-b),$$

and,

$$0 < f(b) - f(a) - m_a(b-a).$$

Adding the two inequalities gives $(m_b - m_a)(b-a) > 0$, which shows that m_b increases strictly in b.

From now on, assume f to be GT, with G above the (unique) tangent at each point. In this case the function m_b must be continuous in b. If we had $m_{b-} = \alpha < m_{b+} = \beta$, then it is straightforward to show that any line $y = f(b) + m(x-b)$ with $\alpha < m < \beta$ can only intersect G at B, and could serve equally as tangent line at B.

The rest follows by standard arguments involving convexity. We have that G is the envelope of the graphs of all tangent lines, i.e,

$$f(x) = \max\{L_b(x) : -\infty < b < \infty\}$$

. The region lying on and above G is thus the intersection of half-planes, and therefore a convex set. From this it follows easily that G lies below its secant lines and, if $a < b < c$,

$$m_a < \frac{f(b)-f(a)}{b-a} < \frac{f(c)-f(a)}{c-a} < \frac{f(c)-f(b)}{c-b} < m_c.$$

From this we obtain that f is differentiable at b with $m_b = f'(b)$ by letting a increase to b and c decrease to b.

Conversely, suppose a function f has a strictly increasing derivative. Then by the Mean Value Theorem we have

$$f(x) > f(b) + f'(b)(x-b), x \neq b,$$

and so the line $y = f(b) + f'(b)(x-b)$ is a geometric tangent line at B. For the uniqueness, suppose we had

$$f(x) > f(b) + m(x-b), x \neq b$$

Letting x decrease to b gives the inequality $f'(b) \geq m$. Letting x increase to b gives the inequality $f'(b) \leq m$. This establishes uniqueness of the geometric tangent line at B and completes the proof.

Let f be a function that has at least one geometric tangent at each point of its graph. Let g be a function such that $g(x)$ is the slope of a geometric tangent at each X on the graph of f. We call such a function g a *version of the derivative of f.*

Theorem 3.2. *Let f have a geometric tangent at each point on its graph. If there is a continuous version of the derivative of f, then f is GT.*

The first part of the proof of Theorem 3.1 shows that any version of the derivative is strictly monotone, say, increasing. Let g be a continuous version. For a given x_0, define a function h by $h(x_0) = m$, where m is the slope of some geometric tangent at x_0, and $h(x) = g(x)$ for all $x \neq x_0$. Then h is also a version of the derivative, so h is increasing. Thus $m < h(x) = g(x)$ for every $x > x_0$. Since g is continuous at x_0, we have $m \leq g(x_0)$ by letting x decrease to x_0. Similarly, $m \geq g(x_0)$. This shows that f has a unique geometric tangent at each x_0 and is therefore a GT function.

References

[1] W.A. Granville, *Elements of the Differential and Integral Calculus*, Ginn and Company, Boston, 1911.

[2] W. Rudin, *Principles of Mathematical Analysis,* 3rd Ed., McGraw-Hill, New York, 1976.

[3] J. Stewart, *Essential Calculus*, Early Transcendentals, 2nd Ed., Brooks/Cole, Belmont, Ca., 2013.